NONLINEAR TENSOR FUNCTIONS OF SEVERAL TENSOR ARGUMENTS

(NELINEINYE TENZORNYE FUNKTSII OT NESKOL'KIKH TENZORNYKH ARGUMENTOV)

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Many fundamental geometrical and physical concepts are represented by scalar or tensor quantities. The mathematical formulation of a wide variety of laws of a geometrical or physical nature is accomplished with the aid of scalar or vector relations. The tensorial expression of equations permits the formulation of laws which are independent of the choice of coordinate systems. Tensor characteristics and tensor equations have additional invariant properties and special peculiarities when the geometric or physical phenomena, objects, laws, and properties admit some symmetry.

Methods are developed below for automatically taking symmetry properties into account both in linear and nonlinear problems by suitable defining parameters which are associated with the basic assumptions in the formulation of the problem under study. Appropriate conclusions are arrived at concerning the effects of symmetry by the use of methods which are analogous to those developed in the closely related theory of similarity and dimensional analysis [1].

The present work is devoted to the solution of two basic problems.

a) It is shown that the properties of textured [oriented] media and crystals can be specified with the aid of tensors. Simple systems of tensors are actually determined as parametric geometrical quantities which define and specify the symmetry properties for all seven types of oriented media and all 32 classes of crystals.

b) The general form is determined for the expression of tensors of arbitrary order when these tensors may be regarded as functions of a system of arguments consisting of a number of scalars and several independent tensors of various orders.

Both problems are intimately related to the consideration of the system of coordinate transformations which generate some symmetry group.

Symmetry problems play a fundamental role in physics. The specialization of the forms of functions and of tensors of various orders which are invariant under suitable symmetry groups is investigated in many works. The appropriate conclusions are applied and have contributed to the discovery of new effects in a number of different applications. A summary of the basic data for different concrete examples is contained in a book by Nye [2]. Detailed references to the earlier literature may be found in the same book.

In algebra a general theory is developed for obtaining and describing the properties of polynomial scalar invariants under finite transformation groups. These polynomials are formed from the components of tensors and vectors. It is shown [3] that for every finite orthogonal group G there always exists an integral rational basis (integrity basis) of invariant polynomials. This integrity basis is a finite number of scalar invariant polynomials formed from the components of given tensors and vectors in such a way that any invariant polynomial formed from the same components can be expressed in terms of them. An integrity basis forms a system of invariants with respect to the finite number of transformations of the group G. It is apparent, however, that its elements, polynomials in the components of given tensors, are not, in general, invariant under any arbitrary coordinate transformation, although such invariants are included in the basis.

The number of elements of an integrity basis, which depends only on the group and on the choice of given tensors and vectors, is generally larger than the number of independent variable components of the given system of tensors and vectors. Therefore, the elements of an integrity basis are, in general, functionally dependent.

The actual construction of an integrity basis for the groups associated with oriented media and crystals has been carried out in works by Döring [4], Smith and Rivlin [5], Pipkin and Rivlin [6], and Sirotin [7,8]. It is shown below that in order to construct tensor functions, it is necessary and sufficient to use a complete system of functionally independent simultaneous invariants [9,10] formed from the components of the tensors which specify the symmetry groups and the other tensor arguments.

The construction of examples of scalars and tensors with specified symmetry is given in papers by Smith and Rivlin [5,6,11,12], in the book by Bhagavantam and Venkatarayudu [13], and in the works of Jahn [14], Shubnikov [15,16,17], and Sirotin [7,8,18,19]. In a paper by Koptsik [20] various tensors of a physical nature are considered. He defines the symmetry of a crystal as "the intersection group of the symmetries of the existing properties of a crystal which are observed at a given instant," (p.935).

Tensors which are functions of tensor arguments are considered in the case of second-order tensors. In this case, functional relations between tensors lead to functional relations between square matrices. In this area the fundamental results reduce to the Cayley-Hamilton formula and to its generalizations to several matrix arguments [21-24, 25-28] (second-order tensors). Basically, however, in these generalizations only polynomial functions of matrices and components of tensors are considered.

1. Fundamental concepts. As is well known, tensors may be regarded as invariant objects which are independent of the choice of the coordinate system and which may be defined by the scalar components in a suitable basis. A tensor basis may be introduced in various ways; in particular, the polyadic product of the base vectors of a coordinate system in some manifold-space can always be taken as a basis.

For simplicity in what follows we shall consider only tensors in three-dimensional space. Let x^1, x^2, x^3 be coordinates of a point of the space and $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ be the vectors of a covariant basis.* We shall denote a tensor of order r by H and its components in the coordinate basis $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ by $H^{\alpha_1 \cdots \alpha_r}$. In this paper we shall use the representation of the tensor H in the form of the sum

$$\boldsymbol{H} = H^{\alpha_1 \ldots \alpha_r} \mathfrak{s}_{\alpha_1} \ldots \mathfrak{s}_{\alpha_r} \tag{1.1}$$

where a summation is understood with respect to all the indices α_1, $\ldots$, α_r, which can take on the values 1, 2, 3. In the general case, the formula (1.1) contains 3^r linearly independent terms, each of which may be considered as a special tensor.

We note that different continuous manifolds and the corresponding different base vectors can be introduced for a single coordinate system. For the same coordinates x^i and the same components $H^{\alpha_1 \cdots \alpha_r}$ it is possible to consider different tensors corresponding to the various bases.

In particular, such manifolds may be considered as different states

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* The coordinate system is arbitrary.

of a given medium having an imbedded Lagrangean coordinate system which moves and deforms with time [26]. Cases are also possible where for a given Lagrangean coordinate system the various manifolds which correspond have different metrics. Thus, it is possible to consider simultaneously different tensors with given components which are the same, but in different bases and in different spaces, some of which may be Euclidean and others non-Euclidean (Kondo, Kröner, Bilby, and others).

The theory below will be developed for tensors in metric spaces.

We shall denote the distance between two points with the coordinates x^i and $x^i + dx^i$ by ds. Let the quantity ds^2 be defined by the formula $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. The matrix $\| g_{ij} \|$ forms the covariant components of the fundamental metric tensor g. The inverse matrix $\| g^{ij} \|$ gives the contravariant components. The contravariant base vectors $\mathfrak{z}^i$ are determined from the formulas $\mathfrak{z}^i = g^{i\alpha}\mathfrak{z}_\alpha$.

The following formulas are valid for the fundamental metric tensor g:

$$\boldsymbol{g} = g_{\alpha\beta}\mathfrak{z}^\alpha\mathfrak{z}^\beta = g^{\alpha\beta}\mathfrak{z}_\alpha\mathfrak{z}_\beta = \delta^\alpha_\beta\mathfrak{z}_\alpha\mathfrak{z}^\beta \quad (\delta^i_j \text{ is the Kronecker delta}) \tag{1.2}$$

Raising and lowering of the scripts of components of the various tensors is accomplished with the aid of the g_{ij} and g^{ij}. The formula (1.1) can be presented in the form:

$$\boldsymbol{H} = \sum_{s=1}^{p} k_s \boldsymbol{H}_s \tag{1.3}$$

where the k_s are scalars and the H_s are certain tensors of order r.

Later we shall always assume that the tensors H_s are linearly independent. It is obvious that $p \leqslant 3^r$.

Let the components of the tensor H be functions of the components of the m tensors

$$\boldsymbol{T}_\varkappa = T_\varkappa^{\alpha_1 \ldots \alpha_{\rho_\varkappa}} \mathfrak{z}_{\alpha_1} \ldots \mathfrak{z}_{\alpha_{\rho_\varkappa}} \qquad (\varkappa = 1, \ldots, m) \tag{1.4}$$

the functions remaining the same, independently of the choice of the coordinate system. The integers $\rho_1, \ldots, \rho_m$ determine the orders of the tensors T_κ. In the general case $\rho_1, \ldots, \rho_m$ are different and are not equal to r. By definition, we then call the tensor H a function of the tensors $T_1, \ldots, T_m$. The tensors T_κ, among which there may be both variable and constant tensors, are the tensor arguments of the tensor function H.

If it is possible to form 3^r linearly independent tensors H_s of order r from the tensors T_K, then in this case the tensor H will satisfy (1.3), in which the scalars k_s will depend only upon the simultaneous invariants of the system of tensors T_K and possibly upon given additional scalar arguments. Below we shall consider only those tensor functions for which the tensor g is included among the tensor arguments T_K.

The tensors H_s can be constructed from the tensors T_K with the aid of two tensor operations: multiplication and contraction. The operation of contraction with respect to any two indices is always possible by virtue of the presence of the tensor g among the tensor arguments. Any multiplication of several tensors leads to a tensor whose order is equal to the sum of the orders of the factors. Contraction with respect to $2l$ indices lowers the order of a tensor by $2l$.

Multiplication and an obvious contraction of a given tensor T having components $T^{ikjl\dots}$ by the tensor S with components $\delta^j_n\delta^i_m$ results in the tensor T^* of the same order with the components

$$T^{*ikjl\dots} = T^{jkil\dots}$$

The tensor T^* is called an isomer of the tensor T. The operation of interchange of indices can be reduced to multiplication by the fundamental tensor and contraction. By definition, a tensor obtained as a result of permutation of several indices is also called an isomer of the tensor T.

Methods are given below to construct general formulas of the type (1.3) for tensor functions. To this end, it is required to construct a linearly independent tensor basis H_s $(s = 1, \dots, p)$ in terms of the tensor arguments (1.4). The construction of the basis H_s from the defining tensors will be accomplished with the aid of the operations of multiplication and contraction.

2. Symmetry groups of tensors. The contravariant components $A^{\alpha_1\dots\alpha_r}$ of the tensor A admit the symmetry group G which is specified by the coordinate transformation matrices*

$$\|a^i_j\| \qquad \left(a^i_j = \frac{\partial y^i}{\partial x^j},\ y^i = y^i(x^j)\right)$$

* For simplicity the enumeration of the elements of the matrices of the group G is omitted, so that $a^{i}_{\cdot j}$ is written instead of $a_{(\nu)}{}^{i}_{\cdot j}$, where $\nu = 1, \dots, h$ and h is the number of elements of the group G.

if for each matrix of the group G the following equation is satisfied:

$$A^{i_1\ldots i_r} = A^{\alpha_1\ldots\alpha_r} a^{i_1}{}_{\alpha_1}\ldots a^{i_r}{}_{\alpha_r} \tag{2.1}$$

If the fundamental tensor g admits a group, the group is called orthogonal. In other words, the transformation matrices of orthogonal groups satisfy the equivalent systems of equations

$$g^{ij} = g^{\alpha\beta}a^{i\cdot}_{\cdot\alpha}a^{j\cdot}_{\cdot\beta}, \qquad g_{ij} = g_{\alpha\beta}a^{\alpha\cdot}_{\cdot i}a^{\beta\cdot}_{\cdot j} \tag{2.2}$$

It is easy to verify that if the group G is orthogonal, then the components of the tensor A having any structure of the indices are invariant* under the coordinate transformations generating the group G provided that the condition (2.1) is met for the contravariant components of A. Therefore, for orthogonal transformations it is possible to speak simply of symmetry of a tensor or of invariance of all its components relative to the group G.

The set of all orthogonal transformations under which a tensor A is invariant forms the symmetry group of the tensor A. The symmetry group of a tensor may consist of only the identity transformation. For an arbitrary second-order tensor (non-symmetrical, $A^{ij} \neq A^{ji}$) the symmetry group consists of two elements: the identity transformation and the transformation of central inversion. For an arbitrary symmetric second-order tensor the symmetry group coincides with the group of self-transformations of a general ellipsoid. If the tensor ellipsoid is an ellipsoid of revolution the symmetry group is infinite. A spherical (isotropic) tensor of second order has a symmetry group which coincides with the full orthogonal group of transformations, just like the fundamental tensor $\boldsymbol{g}$.

Let us consider several tensors $T_1, \ldots, T_m$ and denote their respective symmetry groups by $G_1, \ldots, G_m$. The group G which is formed by the intersection of the groups $G_1, \ldots, G_m$ is called the symmetry group of the set of tensors $T_1, \ldots, T_m$. It is not difficult to see that the tensor $H(T_1, \ldots, T_m)$ will admit the symmetry group G. This follows from the fact that the components of the tensor H are functions of the components of the tensors T_i which are invariant with respect to the group G. Therefore, the components of the tensor H will also be invariant with respect to the group G. In this connection, it is obvious that the symmetry group of a tensor which is obtained as a result of

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* If the group G is not orthogonal it does not follow from (2.1) that the components of the tensor A with another structure of the indices are invariant.

the operations of multiplication and contraction of several tensors will either coincide with the intersection of the symmetry groups of the component tensors or will possess greater symmetry and contain this intersection as a subgroup.

If the tensor H admits the symmetry group G, then the number of linearly independent terms p in the formula (1.3) is, in general, less than 3^r. For a given group G and for a tensor of given order r, the number p can be computed using the theory of group characters [13,14,30]. Tables suitable for the symmetry groups of oriented media and crystals are given in [13,14,18].

If a tensor H of odd order admits only the trivial group G consisting of the identity transformation, the number of terms is $p = 3^r$; in this case the tensor has the most general form. If the tensor H is of even order its symmetry group consists of at least two elements: the identity transformation and central inversion. For symmetry groups consisting of only central inversion and the identity transformation we have $H = 0$ for odd r and, therefore, $p = 0$. For even r we have $p = 3^r$ and, in this case, the tensor of even order has the most general form.

The scalar coefficients k_s in equation (1.3) are, in the general case, functions of the common invariants of the tensors $T_1, \ldots, T_m$ and of any number of given scalars (e.g. temperature, concentration, etc.). Some of the common invariants may be constant parameters, others may be variable. We shall denote the complete system of common invariants [9, 10] of the system of tensors $T_1, \ldots, T_m$ by $\Omega_1, \ldots, \Omega_N$.

It follows from the completeness of the system of invariants that every invariant J formed from the components of the system of tensors $T_1, \ldots, T_m$ satisfies the functional relation:

$$J = f(\Omega_1, \ldots, \Omega_N)$$

By definition, the invariants Ω_i retain their values in their same forms as functions of the components for any of the transformations of coordinates. These invariants can be obtained with the aid of the operations of multiplication and contraction. In this case the invariants are homogeneous polynomials [9,10] in the components of the tensors $T_1, \ldots, T_m$.

Let us assume that among the tensors $T_1, \ldots, T_m$, the tensors $T_\nu, \ldots, T_m$ $(1 < \nu \leqslant m)$ are constant parametric tensors. Let the set of tensors $T_\nu, \ldots, T_m$ admit the finite symmetry group G*. Let us fix the values of the components of the tensors $T_\nu, \ldots, T_m$ given in the coordinate system x^i. After this is done the invariants Ω_i reduce to ω_i, which are functions only of the components of the tensors $T_1, \ldots, T_{\nu-1}$. The equations $\omega_i = \Omega_i$ are true only in the coordinate system x^i; in other coordinate systems these equations are not in general satisfied. However, the equations will be satisfied for all transformations of coordinates determined by the group G*, since for these transformations all the tensors T_ν, ..., T_m are invariant. The quantities ω_i will not in general be invariant under any arbitrary transformation of coordinates. It is clear that some ω_i, those which depend only on the components of the tensors T_ν, ..., T_m or only on the components of the tensors T_1, ..., $T_{\nu-1}$, will not depend on the transformation of coordinates. It is obvious that all the quantities ω_i as functions of the components of the tensors T_1, ..., $T_{\nu-1}$ may be regarded as invariant with respect to the group G*. Thus, the invariant coefficients k_s in formula (1.3) will be functions of the Ω_i. The quantities k_s may be considered as functions of only the invariants ω_i under transformations of coordinates in the group G*.

The invariants ω_i are analogous to the invariants of an integrity basis. The quantities ω_i coincide with an integrity basis for proper choice of the complete system of invariants Ω_i. In the general case variable, functionally independent invariants have special significance. Functionally independent invariants can be selected in various ways.

The actual construction of the tensors H_s in terms of specified defining tensors T_1, ..., T_m is always possible and suitable general methods will be exhibited in examples.

The linear independence of the tensors H_s can be established directly on the basis of geometric considerations or by verification with the aid of the appropriate determinants or by other general methods. In particular, the tensors H_{s_1} and H_{s_2} are linearly independent if they are orthogonal or the symmetry groups of H_{s_1} and H_{s_2} do not coincide, since otherwise these two tensors would be proportional, which would contradict their conditions of symmetry. However, tensors which have the same symmetry group may be linearly independent.

Let the symmetry group G_s correspond to the tensor H_s. In a number of cases it is convenient and advantageous [19] to choose the tensors H_s so that

$$G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots \supseteq G_p$$

It is apparent that it is always possible to take as the first q $(q \leqslant p)$ linearly independent tensors the tensors H_1, ..., H_q, which depend either only on the fundamental tensor $\boldsymbol{g}$ or on $\boldsymbol{g}$ and the third-order tensor $\boldsymbol{E}$.

$$E = |g^{ij}|^{1/2}\,(\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_3 - \mathfrak{d}_1\mathfrak{d}_3\mathfrak{d}_2 + \mathfrak{d}_2\mathfrak{d}_3\mathfrak{d}_1 - \mathfrak{d}_2\mathfrak{d}_1\mathfrak{d}_3 + \mathfrak{d}_3\mathfrak{d}_1\mathfrak{d}_2 - \mathfrak{d}_3\mathfrak{d}_2\mathfrak{d}_1) \qquad (2.3)$$

These tensors correspond to isotropy with respect to the full or the proper orthogonal group. The isotropic tensors $H_1, \ldots, H_q$ of order r are well known from the literature [3,9,10,30]. In three-dimensional space the maximum number q for isotropic tensors of order r equals [30]

$r = 1$	2	3	4	5	6	7	8	9	10
$q = 0$	1	1	3	6	15	36	91	232	603

All isotropic tensors of order r are isomers of the tensor H_1, where

$$H_1^{\alpha_1\ldots\alpha_r} = g^{\alpha_1\alpha_2}\ldots g^{\alpha_r\alpha_{r-1}} \qquad (r = 2k)$$

$$H_1^{\alpha_1\ldots\alpha_r} = E^{\alpha_1\alpha_2\alpha_3} g^{\alpha_4\alpha_5}\ldots g^{\alpha_{r-1}\alpha_r} \qquad (r = 2k+1)$$

The number q is equal to the number of different, linearly independent isomers of the tensor H_1, taking account of the symmetry of the components of the tensor g^{ij}.

If the number r is odd, then $q = 0$ for the full orthogonal group; all tensors of odd order which are invariant under the full orthogonal group reduce to zero. Tensors of odd order which are invariant with respect to the orthogonal group of proper rotations, with $\Delta = |a^i_{\cdot j}| = 1$, can be non-zero only for $r \geqslant 3$. For $r = 3$, we have $H_1 = E$ and, therefore, $q = 1$.

The presence of symmetry of tensor functions with respect to some group of permutations of the indices will, generally speaking, decrease the numbers p and q. Formulas for tensor functions having certain symmetries with respect to some indices are always easily obtained from the general formulas by using the operations of symmetrization and alternation on the proper indices, retaining in the process only the linearly independent terms.

3. Tensors which specify the geometric symmetry of oriented media and crystals [29]. A medium is called isotropic if all its properties at each point are invariant under the group of orthogonal transformations. We can distinguish between the following two types of isotropic media:

1) isotropic media with respect to the full orthogonal group of coordinate transformations with $\Delta = \pm 1$,

2) isotropic media with respect to the group of rotations with $\Delta = +1$ (gyrotropic media).

It is easy to see that in the first case the symmetry properties are completely characterized by the fundamental tensor $\boldsymbol{g}$. The condition of invariance of the components of the tensor g can be considered as the condition which defines the infinite class of all real matrices which are elements of the full orthogonal group.

The group of rotations with $\Delta = +1$, which defines gyrotropic media, is a subgroup of the full orthogonal group. This subgroup can be singled out by supplementing the condition (2.2) with the additional requirement of invariance of the components of the tensor $\boldsymbol{E}$ defined by formula (2.3). Therefore, the infinite set of elements of the group of rotations is determined completely by the condition of invariance of the tensors g and $\boldsymbol{E}$. These two tensors may be considered as the tensors which determine the group of rotations with $\Delta = +1$.

Later we shall use the abbreviated symbols proposed by Shubnikov [15, 16] as notation for symmetry groups. According to these rules, the full orthogonal group is denoted by the symbol $\infty/\infty \times m$ (the generating elements of the group are: intersecting axes of infinite order and a reflecting plane of symmetry m). The group of rotations corresponds to the symbol ∞/∞.

Results are given in Section 2 on the general form of tensor functions for tensors of any order when isotropy is present, i.e. when the arguments are only $\boldsymbol{g}$ or $\boldsymbol{g}$ and $\boldsymbol{E}$.

The simplest example of an anisotropic medium is the oriented medium. We shall call a medium an oriented [textured] medium if all its properties at each point are invariant under an infinite orthogonal group containing rotations of arbitrary angle about some axis. Obviously, the symmetry groups of oriented media are subgroups of the full orthogonal group.

A simple analysis shows that only seven different types of oriented media are possible, including the two types of isotropic media. The appropriate geometric illustrations for the different types of oriented media and the corresponding tensors and vectors which specify the symmetry groups of the oriented media are given in the table below. The correctness of these results is easily verified directly.

An anisotropic medium with a continuous or discrete structure is called a crystal if it is possible to introduce a system of triply periodic Bravais lattices (with the same periods in the various lattices in a fixed coordinate system) having the same geometric properties as the medium under consideration. The set of Bravais lattices with given periods can admit finite point symmetry groups. The form of these groups depends on the structure of the set of lattices being examined and on the elementary parallelepiped of periods.

As is well known [2,16], there are only 32 different symmetry classes of crystals described by finite point groups. In the table (pp. 608, 609) the characteristic data are presented for all 32 crystal classes; the corresponding geometric figures illustrate each symmetry group.

The unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form the orthogonal crystallographic basis. The orientation of this basis relative to the figure of symmetry of the crystal is indicated in the sketch. At the upper left of each box the notation of the corresponding group according to Shubnikov is given. Moreover, each box contains symbols we have used for a set of simple tensors which characterize and specify the given group. The definitions of these tensors are given by formulas which also appear in the same table.*

Let us consider the tensors which determine the symmetries of the groups of the cubic system. We shall prove that the tensor O_h is invariant under a group of 48 transformations which give an isomorphic representation of the group $\bar{6}/4$, and that there are no other transformations under which the tensor O_h is invariant. To carry out the proof let us find all real transformations under which the tensor O_h is invariant.

The condition of invariance of the contravariant components of the tensor O_h are equivalent to the following system of nonlinear algebraic equations for the nine elements of the matrix of transformation $\| a^i{}_j \|$

$$a^\alpha{}_1 a^\beta{}_1 a^\gamma{}_1 a^\delta{}_1 + a^\alpha{}_2 a^\beta{}_2 a^\gamma{}_2 a^\delta{}_2 + a^\alpha{}_3 a^\beta{}_3 a^\gamma{}_3 a^\delta{}_3 = \begin{cases} 1 \\ 0 \end{cases} \tag{3.1}$$

The right-hand side is to be set equal to unity if $\alpha = \beta = \gamma = \delta$ and to zero in the remaining cases. Now setting $\alpha = \beta$ and $\gamma = \delta$ for $\alpha \neq \gamma$, we obtain the equations:

$$(a^\alpha{}_1)^2 (a^\gamma{}_1)^2 + (a^\alpha{}_2)^2 (a^\gamma{}_2)^2 + (a^\alpha{}_3)^2 (a^\gamma{}_3)^2 = 0 \qquad (\alpha \neq \gamma) \tag{3.2}$$

It follows from (3.2) that

$$a^\alpha{}_i a^\gamma{}_i = 0 \tag{3.3}$$

Since the determinant $|a^\alpha{}_i| \neq 0$, we conclude from (3.3) that there is only one non-zero element in each column and in each row of the matrix $\| a^\alpha{}_i \|$.

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* In this table and in what follows, powers of vectors are to be understood as dyadic or polyadic products.

x^1, x^2, x^3 — are the crystallographic Cartesian coordinates

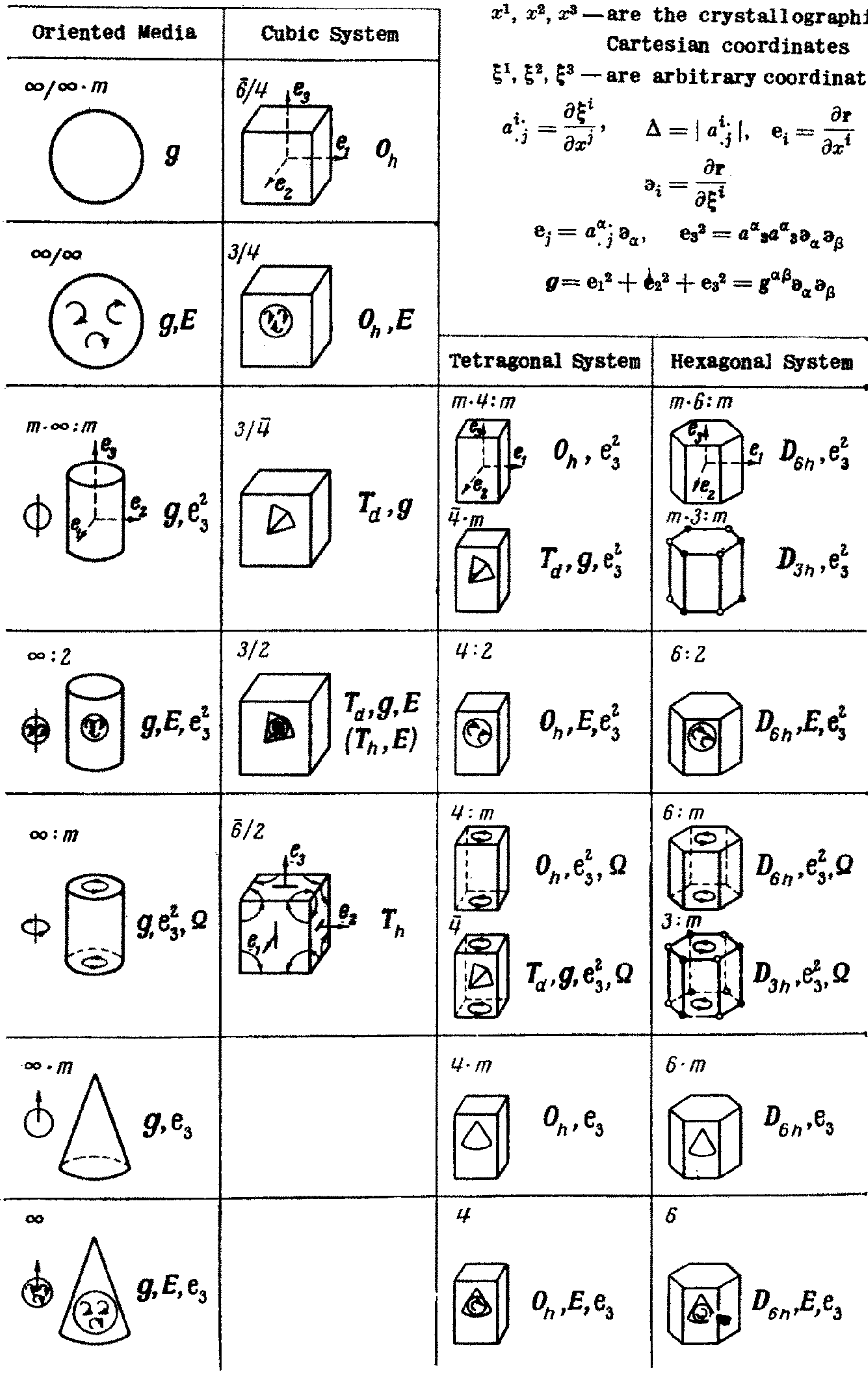
ξ^1, ξ^2, ξ^3 — are arbitrary coordinates

$$a^{i\cdot}_{\cdot j} = \frac{\partial \xi^i}{\partial x^j}, \quad \Delta = |a^{i\cdot}_{\cdot j}|, \quad \mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i}$$

$$\mathbf{э}_i = \frac{\partial \mathbf{r}}{\partial \xi^i}$$

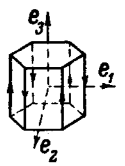
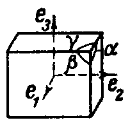
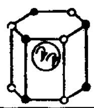
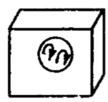
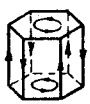
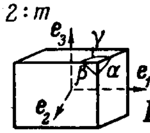
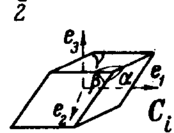
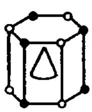
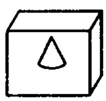
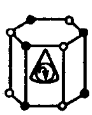
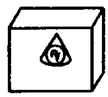
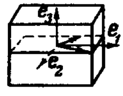
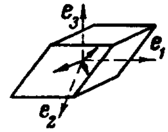
$$\mathbf{e}_j = a^{\alpha}_{\cdot j}\mathbf{э}_\alpha, \quad \mathbf{e}_3^2 = a^{\alpha}_{\cdot 3}a^{\alpha}_{\cdot 3}\mathbf{э}_\alpha \mathbf{э}_\beta$$

$$g = \mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2 = g^{\alpha\beta}\mathbf{э}_\alpha \mathbf{э}_\beta$$



Oriented Media	Cubic System	Tetragonal System	Hexagonal System
$\infty/\infty\cdot m$: g	$\bar{6}/4$: O_h		
∞/∞: g, E	$3/4$: O_h, E		
$m\cdot\infty:m$: $g, \mathbf{e}_3^2$	$3/\bar{4}$: T_d, g	$m\cdot 4:m$: $O_h, \mathbf{e}_3^2$; $\bar{4}\cdot m$: $T_d, g, \mathbf{e}_3^2$	$m\cdot 6:m$: $D_{6h}, \mathbf{e}_3^2$; $m\cdot 3:m$: $D_{3h}, \mathbf{e}_3^2$
$\infty:2$: $g, E, \mathbf{e}_3^2$	$3/2$: T_d, g, E (T_h, E)	$4:2$: $O_h, E, \mathbf{e}_3^2$	$6:2$: $D_{6h}, E, \mathbf{e}_3^2$
$\infty:m$: $g, \mathbf{e}_3^2, \Omega$	$\bar{6}/2$: T_h	$4:m$: $O_h, \mathbf{e}_3^2, \Omega$; $\bar{4}$: $T_d, g, \mathbf{e}_3^2, \Omega$	$6:m$: $D_{6h}, \mathbf{e}_3^2, \Omega$; $3:m$: $D_{3h}, \mathbf{e}_3^2, \Omega$
$\infty\cdot m$: $g, \mathbf{e}_3$		$4\cdot m$: $O_h, \mathbf{e}_3$	$6\cdot m$: $D_{6h}, \mathbf{e}_3$
∞: $g, E, \mathbf{e}_3$		4: $O_h, E, \mathbf{e}_3$	6: $D_{6h}, E, \mathbf{e}_3$

$$E = e_1e_2e_3 - e_2e_1e_3 + e_2e_3e_1 - e_3e_2e_1 + e_3e_1e_2 - e_1e_3e_2 =$$
$$= \Delta\,(\partial_1\partial_2\partial_3 - \partial_2\partial_1\partial_3 + \partial_2\partial_3\partial_1 - \partial_3\partial_2\partial_1 + \partial_3\partial_1\partial_2 - \partial_1\partial_3\partial_2)$$
$$\Omega = e_1e_2 - e_2e_1 = (a^{\alpha}{}_1 a^{\beta}{}_2 - a^{\alpha}{}_2 a^{\beta}{}_1)\,\partial_\alpha\partial_\beta = a^{\alpha}{}_1 a^{\beta}{}_2\,(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)$$
$$O_h = e_1^4 + e_2^4 + e_3^4 = (a^{\alpha}{}_1 a^{\beta}{}_1 a^{\gamma}{}_1 a^{\delta}{}_1 + a^{\alpha}{}_2 a^{\beta}{}_2 a^{\gamma}{}_2 a^{\delta}{}_2 + a^{\alpha}{}_3 a^{\beta}{}_3 a^{\gamma}{}_3 a^{\delta}{}_3)\,\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta$$
$$T_h = e_1^2e_2^2 + e_2^2e_3^2 + e_3^2e_1^2, \quad T_d = e_1e_2e_3 + e_2e_1e_3 + e_2e_3e_1 + e_3e_2e_1 + e_3e_1e_2 + e_1e_3e_2$$
$$D_{3h} = e_1^3 - e_1e_2^2 - e_2e_1e_2 - e_2^2e_1, \quad D_{3d} = e_3\,(e_1^3 - e_1e_2^2 - e_2e_1e_2 - e_2^2e_1)$$
$$D_{6h} = (e_1^3 - e_1e_2^2 - e_2e_1e_2 - e_2^2e_1)^2, \qquad D_{2h} = \lambda^{11}e_1^2 + \lambda^{22}e_2^2 + \lambda^{33}e_3^2 =$$
$$= \lambda^{ij} a^{\alpha}_{\cdot i} a^{\beta}_{\cdot j}\,\partial_\alpha\partial_\beta = d^{\alpha\beta}\partial_\alpha\partial_\beta \quad (\lambda^{11} \neq \lambda^{22} \neq \lambda^{33} \neq \lambda^{11} \neq 0,\ d^{\alpha\beta} = d^{\beta\alpha})$$
$$C_i = D_{2h} + \omega^{ij}e_ie_j = C^{\alpha\beta}\partial_\alpha\partial_\beta; \quad \omega^{ij} = -\omega^{ji} \neq 0$$

Trigonal System	Rhombic System	Monoclinic System	Triclinic System
$\bar{6}\cdot m$  D_{3d}, e_3^2	$m\cdot 2:m$  D_{2h} $\alpha=\beta=\gamma=90^\circ$		
$3:2$  D_{3h}, E, e_3^2	$2:2$  D_{2h}, E		
$\bar{6}$  D_{3d}, e_3^2, Ω		$2:m$  D_{2h}, Ω $\alpha=\beta=90^\circ$ $\gamma\neq 90^\circ$	$\bar{2}$  C_i $\alpha\neq\beta\neq\gamma\neq\alpha$ $\alpha,\beta,\gamma\neq 90^\circ$
$3\cdot m$  D_{3h}, e_3	$2\cdot m$  D_{2h}, e_3		
3  D_{3h}, E, e_3		2  D_{2h}, E, e_3 m  D_{2h}, e_1, e_2	1  e_1, e_2, e_3

Since $(a^{\alpha}{}_1)^4 + (a^{\alpha}{}_2)^4 + (a^{\alpha}{}_3)^4 = 1$ for $\alpha = \beta = \gamma = \delta$, in accordance with (3.1), the following equality holds for each real non-zero element of the matrix $\| a^{\alpha}{}_i \|$

$$a^p{}_q = \pm 1 \tag{3.4}$$

Enumeration of all possible cases of (3.3) and (3.4) shows that the matrices consisting of the elements $(a^p{}_q)^2$, which are either equal to 1 or 0, can have the following forms:

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{Vmatrix} \tag{3.5}$$

A system consisting of only six matrices has been obtained. If, in accordance with (3.4), account is taken of the possibilities of different signs for the $a^p{}_q$, then each of the matrices (3.5) generates eight matrices for $\| a^p{}_q \|$. For example, the matrices corresponding to the first matrix of (3.5) are:

$$\begin{gathered} \begin{Vmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{Vmatrix}, \quad \begin{Vmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \quad \begin{Vmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{Vmatrix}, \quad \begin{Vmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{Vmatrix} \\ \begin{Vmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{Vmatrix}, \quad \begin{Vmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \quad \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{Vmatrix}, \quad \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{Vmatrix} \end{gathered} \tag{3.6}$$

As is known, by the definition of the symmetry group of the cube $\bar{6}/4$, the system of matrices of the type (3.6) for each matrix of the system (3.5) forms the complete group of transformation matrices for symmetry of the cube of the group $\bar{6}/4$, and consists of $6 \times 8 = 48$ orthogonal matrices. Thus, every matrix corresponding to a solution of the system of equations (3.1) must be one of the 48 matrices of the system (3.6). On the other hand, it is easy to assure oneself that the converse proposition is also true: each matrix of the system of 48 matrices which has been found provides a solution of the system of equations (3.1).

Let us now find the matrices of the group of transformations under which the tensor T_d is invariant. The conditions of invariance of the contravariant components of the tensor T_d are equivalent to the following system of nonlinear algebraic equations for the nine elements of the transformation matrix, $a^i{}_j$

$$a^{\alpha}{}_1 a^{\beta}{}_2 a^{\gamma}{}_3 + a^{\alpha}{}_2 a^{\beta}{}_1 a^{\gamma}{}_3 + a^{\alpha}{}_3 a^{\beta}{}_2 a^{\gamma}{}_1 + a^{\alpha}{}_1 a^{\beta}{}_3 a^{\gamma}{}_2 + a^{\alpha}{}_2 a^{\beta}{}_3 a^{\gamma}{}_1 + a^{\alpha}{}_3 a^{\beta}{}_1 a^{\gamma}{}_2 = \begin{cases} 1 \\ 0 \end{cases} \tag{3.7}$$

The right side of (3.7) should be set equal to unity if α, β, γ are all different and to zero if at least one pair of α, β, γ are the same.

Let us take the equations of (3.7) for which $\gamma = \beta$. These equations have the form

$$a^{\alpha}{}_1 a^{\beta}{}_2 a^{\beta}{}_3 + a^{\alpha}{}_2 a^{\beta}{}_1 a^{\beta}{}_3 + a^{\alpha}{}_3 a^{\beta}{}_1 a^{\beta}{}_2 = 0 \qquad \begin{pmatrix}\alpha = 1, 2, 3 \\ \beta = 1, 2, 3\end{pmatrix} \tag{3.8}$$

Since $|a^i{}_j| \neq 0$, it follows from the system of equations (3.8) that

$$a^{\beta}{}_i a^{\beta}{}_j = 0 \tag{3.9}$$

where β is any fixed index. From (3.9) and the condition that $|a^i{}_j| \neq 0$ it may be concluded that there is only one non-zero element in each row and each column of the matrix $\| a^i{}_j \|$. There are only six such matrices having different structures of the indices on the non-zero elements:

$$\begin{Vmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{Vmatrix}, \quad \begin{Vmatrix} a_2 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & c_2 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & a_3 & 0 \\ b_3 & 0 & 0 \\ 0 & 0 & c_3 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & a_4 & 0 \\ 0 & 0 & b_4 \\ c_4 & 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & a_5 \\ 0 & b_5 & 0 \\ c_5 & 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & a_6 \\ b_6 & 0 & 0 \\ 0 & c_6 & 0 \end{Vmatrix} \tag{3.10}$$

Equations (3.7) with different indices α, β, γ give:

$$a_i b_i c_i = 1 \qquad (i = 1, \ldots, 6) \tag{3.11}$$

It may easily be seen that for orthogonal transformations, when the conditions

$$\sum_{\alpha=1}^{3} a^i{}_{\alpha}\, a^j{}_{\alpha} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{3.12}$$

are satisfied, the following equalities hold:

$$a_i = \pm 1, \qquad b_i = \pm 1, \qquad c_i = \pm 1 \tag{3.13}$$

In the general case, in order to obtain a representation of the symmetry group $3/\bar{4}$, the requirement of the invariance of the tensor T_d must be supplemented by the condition of invariance of the tensor g, since only in this case will the conditions (3.12) which form part of the definition of crystal symmetry groups be satisfied.*

---

* It is easy to verify that for $|\mathbf{e}_i| = 1$, the equation $2g = T_d : T_d$ holds, where the contraction is carried out with respect to two similarly located indices. However, it does not follow from this equation that g is invariant under the transformation (3.10) (with (3.11)).

The system of matrices (3.10) together with the conditions (3.13) determines the 48 matrices of the symmetry group $\overline{6}/4$. However, the additional equalities (3.11) select a subgroup of 24 matrices for which either $a_i = b_i = c_i = 1$ or two elements of the three numbers a_i, b_i, c_i are equal to -1. For instance, we obtain only four matrices from the first one of (3.10)

$$\begin{Vmatrix}+1 & 0 & 0\\ 0 & +1 & 0\\ 0 & 0 & +1\end{Vmatrix}, \quad \begin{Vmatrix}+1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{Vmatrix}, \quad \begin{Vmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & +1\end{Vmatrix}, \quad \begin{Vmatrix}-1 & 0 & 0\\ 0 & +1 & 0\\ 0 & 0 & -1\end{Vmatrix} \tag{3.14}$$

It is easy to verify that the system of 24 matrices which has been found and which represents the group $3/\overline{4}$ is the solution of the equations (3.7) provided that the matrices sought are orthogonal.

Let us now consider the conditions of invariance of the tensor T_h. The following system of equations for $a^i{}_j$, the elements of the transformation matrix, is equivalent to the condition of invariance of the contravariant components of the tensor T_h:

$$a^\alpha{}_2 a^\beta{}_2 a^\gamma{}_3 a^\delta{}_3 + a^\alpha{}_3 a^\beta{}_3 a^\gamma{}_1 a^\delta{}_1 + a^\alpha{}_1 a^\beta{}_1 a^\gamma{}_2 a^\delta{}_2 = \begin{cases}1\\0\end{cases} \tag{3.15}$$

where the right side should be set equal to 1 for $\alpha = \beta = 2$, $\gamma = \delta = 3$; $\alpha = \beta = 3$, $\gamma = \delta = 1$; $\alpha = \beta = 1$, $\gamma = \delta = 2$ and to zero in all other cases. From (3.15) we have

for $\alpha = \beta = 1$, $\gamma = \delta = 1$, 3

$$a^1{}_2 a^1{}_3 = 0, \quad a^1{}_2 a^3{}_3 = 0, \quad a^1{}_3 a^1{}_1 = 0, \quad a^1{}_3 a^3{}_1 = 0, \quad a^1{}_1 a^1{}_2 = 0, \quad a^1{}_1 a^3{}_2 = 0 \tag{3.16}$$

for $\alpha = \beta = 2$, $\gamma = \delta = 1$, 2

$$a^2{}_2 a^1{}_3 = 0, \quad a^2{}_2 a^2{}_3 = 0, \quad a^2{}_3 a^1{}_1 = 0, \quad a^2{}_3 a^2{}_1 = 0, \quad a^2{}_1 a^1{}_2 = 0, \quad a^2{}_1 a^2{}_2 = 0 \tag{3.17}$$

for $\alpha = \beta = 3$, $\gamma = \delta = 2$, 3

$$a^3{}_2 a^2{}_3 = 0, \quad a^3{}_2 a^3{}_3 = 0, \quad a^3{}_3 a^2{}_1 = 0, \quad a^3{}_3 a^3{}_1 = 0, \quad a^3{}_1 a^2{}_2 = 0, \quad a^3{}_1 a^3{}_2 = 0 \tag{3.18}$$

It follows from the 18 equations (3.16) to (3.18) and from the condition that $|a^i{}_j| \neq 0$ that only one element in each row and each column of the matrix $\|a^i{}_j\|$ can be different from zero. If

$$a^1{}_1 \neq 0, \text{ then } a^1{}_2 = a^1{}_3 = a^3{}_2 = a^2{}_3 = a^2{}_1 = a^3{}_1 = 0$$

Thus we obtain the matrices

$$\text{for } a^1{}_1 \neq 0 \qquad\qquad \text{for } a^1{}_2 \neq 0 \qquad\qquad \text{for } a^1{}_3 \neq 0$$

$$\begin{Vmatrix} a^1{}_1 & 0 & 0 \\ 0 & a^2{}_2 & 0 \\ 0 & 0 & a^3{}_3 \end{Vmatrix} \qquad \begin{Vmatrix} 0 & a^1{}_2 & 0 \\ 0 & 0 & a^2{}_3 \\ a^3{}_1 & 0 & 0 \end{Vmatrix} \qquad \begin{Vmatrix} 0 & 0 & a^1{}_3 \\ a^2{}_1 & 0 & 0 \\ 0 & a^3{}_2 & 0 \end{Vmatrix} \tag{3.19}$$

The three equations (3.15), when the right side is equal to unity and $a^1{}_1 \neq 0$ result in

$$(a^2{}_2)^2\,(a^3{}_3)^2 = 1, \qquad (a^3{}_3)^2\,(a^1{}_1)^2 = 1, \qquad (a^1{}_1)^2\,(a^2{}_2)^2 = 1 \tag{3.20}$$

The real solutions of these equations and the equations which are obtained analogously for $a^1{}_2 \neq 0$ and $a^1{}_3 \neq 0$ are given by the equalities

$$\begin{aligned} a^1{}_1 &= \pm 1, & a^2{}_2 &= \pm 1, & a^3{}_3 &= \pm 1 \\ a^1{}_2 &= \pm 1, & a^2{}_3 &= \pm 1, & a^3{}_1 &= \pm 1 \\ a^1{}_3 &= \pm 1, & a^2{}_1 &= \pm 1, & a^3{}_2 &= \pm 1 \end{aligned} \tag{3.21}$$

It follows from the values of $a^i{}_j$ which have been found that each of the matrices (3.19) splits up into eight matrices. We obtain a subgroup of the group of matrices $\bar{6}/4$, the subgroup consisting, all told, $3 \times 8 = 24$ orthogonal matrices. It is clear that the solution obtained satisfies the entire system of equations (3.15) and that every real solution is contained in the one just found.

The addition of the tensor E as a defining quantity results in the exclusion of matrices with $\Delta = -1$, since E is invariant only with respect to the group of proper rotations, for $\Delta = +1$.

The set of two tensors O_h and E singles out a subgroup consisting of the 24 matrices with $\Delta = +1$ from the group of 48 matrices found for O_h. The set of tensors g, T_d, E also specifies a subgroup consisting of 12 matrices with $\Delta = +1$ from the 24 matrices which were found for the group of g, T_d. By actually singling out the proper matrices one can show that the transformation groups corresponding to the system of 12 matrices for the tensors g, T_d, E and that for the tensors T_h, E coincide.

The equivalence of the tensors and the corresponding symmetry groups for the tetragonal system which are indicated in the table is a consequence of the following considerations. The symmetry group of the tetragonal system can be obtained as the intersection of the corresponding symmetry groups of crystals of the cubic system and symmetry groups of oriented media. Therefore, the specification of the proper subgroups from the groups of the cubic system and of the oriented media may be accomplished by forming sets of tensors from the tensors which specify the corresponding cubic symmetry groups and those which specify groups for the oriented media. It is easy to see directly that the conditions of invariance indicated by the sets of tensors for each of the seven classes of the tetragonal system determines the group of transformation matrices of the corresponding symmetry group of these crystal classes.

In order to justify the choice of the tensors which specify the symmetries of the hexagonal and trigonal systems, we must consider the conditions of invariance of the components of the following pairs of tensors D_{6h} and $\mathbf{e}_3{}^2$, D_{3h} and $\mathbf{e}_3{}^2$, D_{3d} and $\mathbf{e}_3{}^2$. The conditions of invariance of the dyad $\mathbf{e}_3{}^2$ selects only the matrices of the following form

$$\begin{Vmatrix} a^1{}_1 & a^1{}_2 & a^1{}_3 \\ a^2{}_1 & a^2{}_2 & a^2{}_3 \\ 0 & 0 & \pm 1 \end{Vmatrix} \tag{3.22}$$

as admissible coordinate transformation matrices. It follows from the invariance of D_{6h} or D_{3h} or D_{3d} that $a^1{}_3 = a^2{}_3 = 0$. If we require the invariance of the vector $\mathbf{e}_3$ instead of $\mathbf{e}_3{}^2$, we are led to transformation matrices of the form

$$\begin{Vmatrix} a^1{}_1 & a^1{}_2 & 0 \\ a^2{}_1 & a^2{}_2 & 0 \\ 0 & 0 & +1 \end{Vmatrix} \tag{3.23}$$

Since D_{6h}, D_{3h} and D_{3d} are expressed in terms of the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ only, the invariance of these tensors is related to the structure of the second-order matrices:

$$D = \begin{Vmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{Vmatrix} \tag{3.24}$$

In order to determine the structures of the matrices D, it is convenient to introduce a complex basis by means of the formulas

$$\mathbf{j}_1 = \mathbf{e}_1 + i\mathbf{e}_2, \qquad \mathbf{j}_2 = \mathbf{e}_1 - i\mathbf{e}_2$$

In this basis the tensors D_{3h}, D_{6h} and D_{3d} take the form:

$$2\boldsymbol{D}_{3h} = \mathbf{j}_1{}^3 + \mathbf{j}_2{}^3, \qquad 4\boldsymbol{D}_{6h} = (\mathbf{j}_1{}^3 + \mathbf{j}_2{}^3)^2, \qquad 2\boldsymbol{D}_{3d} = \mathbf{e}_3\,(\mathbf{j}_1{}^3 + \mathbf{j}_2{}^3)$$

The conditions of invariance of these tensors in the real basis can be rewritten as conditions of invariance in the complex basis. If the transformation formulas of the complex basis have the form

$$\mathbf{j}_i = b^\alpha{}_i \mathbf{j}'_\alpha$$

then the relation between the matrices $\| a^i{}_j \|$ and $\| b^i{}_j \|$ is determined by the equations

$$\begin{Vmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{Vmatrix} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{Vmatrix} \times \begin{Vmatrix} b^1{}_1 & b^1{}_2 \\ b^2{}_1 & b^2{}_2 \end{Vmatrix} \times \begin{Vmatrix} 1 & i \\ 1 & -i \end{Vmatrix} \tag{3.25}$$

The condition of invariance of the tensor D_{3d} leads to the following system of equations for the $b^i{}_j$

$$b^\alpha{}_1 b^\beta{}_1 b^\gamma{}_1 + b^\alpha{}_2 b^\beta{}_2 b^\gamma{}_2 = \begin{cases} 1 & \text{for } \alpha = \beta = \gamma \\ 0 & \text{in the remaining cases.} \end{cases}$$

In expanded form this system is equivalent to the equations:

$$\begin{aligned} &(b^1{}_1)^3 + (b^1{}_2)^3 = 1, \qquad b^1{}_1 (b^2{}_1)^2 + b^1{}_2 (b^2{}_2)^2 = 0 \\ &(b^2{}_1)^3 + (b^2{}_2)^3 = 1, \qquad b^2{}_1 (b^1{}_1)^2 + b^2{}_2 (b^1{}_2)^2 = 0 \end{aligned} \tag{3.26}$$

All solutions of (3.26) satisfying the condition $|b^i{}_j| \neq 0$ are easily found from these equations. Since the $a^i{}_j$ are real, it follows from (3.25) that $b^1{}_1 = \bar{b}^2{}_2$ and $b^1{}_2 = \bar{b}^2{}_1$.

Taking this into account, we obtain six matrices for $\| b^i{}_j \|$:

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} \varepsilon & 0 \\ 0 & \varepsilon^2 \end{Vmatrix}, \quad \begin{Vmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & \varepsilon \\ \varepsilon^2 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & \varepsilon^2 \\ \varepsilon & 0 \end{Vmatrix} \quad \left(\varepsilon = \exp\frac{2\pi i}{3}\right) \tag{3.27}$$

The orthogonality of the corresponding matrices (3.22) is obtained automatically.

By use of formulas (3.27), (3.25) and (3.22) it is easy to write out the 12 matrices corresponding to invariance of the tensors D_{3h}, $\mathbf{e}_3{}^2$ which characterize the class $m \times 3 : m$ of the hexagonal system. The invariance of D_{3h}, $\mathbf{e}_3$ determines six matrices obtained from (3.23), (3.25) and (3.27) corresponding to the class $3 \times m$ of the trigonal system.

The conditions of invariance of D_{3d} and $\mathbf{e}_3{}^2$ modify equations (3.26) somewhat. The solution of the corresponding equations leads to a system of twelve matrices. The first six of these, which correspond to the invariance of $\mathbf{e}_3$, coincide with the matrices of the class $3 \times m(D_{3h}, \mathbf{e}_3)$, and the other six are obtained from the first ones by changing the signs of all the components of the matrices. The conditions of invariance of D_{6h} and $\mathbf{e}_3{}^2$ lead to matrices of the type (3.22). In the corresponding equations of type (3.26), ±1 must be written instead of +1. As a consequence of this, the corresponding solution contains the twelve matrices of the class $m \times 3 : m$ and, in addition, the following twelve matrices:

$$\begin{Vmatrix} \tau^3 & 0 & 0 \\ 0 & \tau^3 & 0 \\ 0 & 0 & \pm 1 \end{Vmatrix}, \quad \begin{Vmatrix} \tau & 0 & 0 \\ 0 & \tau^5 & 0 \\ 0 & 0 & \pm 1 \end{Vmatrix}, \quad \begin{Vmatrix} \tau^5 & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \pm 1 \end{Vmatrix}$$

$$\begin{Vmatrix} 0 & \tau^3 & 0 \\ \tau^3 & 0 & 0 \\ 0 & 0 & \pm 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & \tau & 0 \\ \tau^5 & 0 & 0 \\ 0 & 0 & \pm 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & \tau^5 & 0 \\ \tau & 0 & 0 \\ 0 & 0 & \pm 1 \end{Vmatrix} \qquad \left(\tau = \exp\frac{\pi i}{3}\right)$$

The corresponding real matrices are easily written out with the aid of formula (3.25).

The tensor parameters for all the remaining classes of the hexagonal and trigonal systems are easily obtained by considering the intersections of suitable groups whose tensor characteristics have already been established. The reason for this is that the symmetry groups of these classes are subgroups of the symmetry groups which have been investigated above.

As for the rhombic, monoclinic, and triclinic systems, the tensorial characteristics indicated in the table are immediately apparent. It is clear that the corresponding sets of tensors which specify the symmetry groups are not uniquely determined.

In each case, another system of tensors having a one-to-one relation with the system given in the table may replace the latter. In particular, the number and powers of the tensors in a system need not remain the same. For example, instead of the tensors indicated in the table, the following correspondence of tensors and groups may be used:*

$m \cdot 2 : m$	$\mathbf{e}_1^2, \mathbf{e}_2^2, \mathbf{e}_3^2,$	2	$\mathbf{e}_1^2, \mathbf{e}_2^2, \mathbf{e}_3, \boldsymbol{E}$
$2 : 2$	$\mathbf{e}_1^2, \mathbf{e}_2^2, \mathbf{e}_3^2, \boldsymbol{E},$	m	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3^2$
$2 \cdot m$	$\mathbf{e}_1^2, \mathbf{e}_2^2, \mathbf{e}_3,$	$\overline{2}$	$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$
$2 : m$	$\mathbf{e}_1^2, \mathbf{e}_2^2, \mathbf{e}_3^2, \boldsymbol{\Omega},$		

Each tensor of this system can easily be expressed in terms of the tensors given in the table. The inverse relations are immediately obvious.

The problem of determination of tensors which specify the symmetry groups of crystals and oriented media has been considered above. The inverse problem of determination of the orthogonal symmetry groups corresponding to a given tensor has been solved in important special cases.

* Products and powers of vectors are to be understood as dyadic products.

4. Tensor functions of tensors characterizing the geometric properties of oriented media and crystals. General formulas of the type (1.3) are given below which are valid in arbitrary coordinates for the components of vectors A^i, second-order tensors A^{ij}, third-order tensors A^{ijk}, and fourth-order tensors A^{ijkl} for oriented media* and crystals. These tensors are functions of the tensor arguments in the table which determine the various symmetry groups.

Since the simultaneous invariants of the tensors which determine the symmetry groups are absolute constants, the invariant coefficients k_s ($s = 1, \ldots, p$) are numerical constants or functions of certain scalars which may also be present in the list of defining quantities in addition to the specifying tensors.

Only p linearly independent terms are written out in the formulas. The choice of the terms may be changed; but in every other case a proper choice of terms can be represented as linear combinations of the terms written out in the formulas.

The problem of selection of linearly independent tensors may prove to be important when using various supplementary hypotheses about the character of the functional relations (linear dependence on certain components, etc.).

The known results** when the following symmetry conditions are used:

$$A^{ij} = A^{ji}, \quad A^{ijk} = A^{ikj}, \quad A^{ijkl} = A^{ijlk}, \quad A^{ijkl} = A^{jikl}, \quad A^{ijkl} = A^{klij}$$

are easy to obtain from the formulas given. The above conditions are fulfilled when additional limitations are imposed on the invariant coefficients. Proper formulas are obtained from the ones presented by means of the operation of symmetrization.

Oriented media

Class $\infty/\infty\cdot m$ $(\boldsymbol{g})$

$$A^i = 0, \quad A^{ij} = kg^{ij}, \quad A^{ijk} = 0, \quad A^{ijkl} = k_1 g^{ij} g^{kl} + k_2 g^{ik} g^{jl} + k_3 g^{il} g^{jk}$$

Class ∞/∞ $(\boldsymbol{g}, \boldsymbol{E})$

$$A^i = 0, \quad A^{ij} = kg^{ij}, \quad A^{ijk} = kE^{ijk}, \quad A^{ijkl} = A^{ijkl}\ (\infty/\infty\cdot m)$$

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* Analogous formulas containing errors were published in [28]. The corrected formulas are given here.

** (See preceding footnote)

Class $m\cdot\infty : m$ $(g,\ B = e_3^2)$

$$A^i = 0,\quad A^{ij} = k_1 g^{ij} + k_2 B^{ij},\quad A^{ijk} = 0,\quad A^{ijkl} = A^{ijkl}\ (\infty / \infty\cdot m) + k_4 g^{ij} B^{kl} + k_5 g^{ik} B^{jl} + k_6 g^{il} B^{jk} + k_7 g^{kl} B^{ij} + k_8 g^{jl} B^{ik} + k_9 g^{jk} B^{il} + k_{10} B^{ij} B^{kl}$$

Class $\infty : 2$ $(g,\ B = e_3^2,\ E)$

$$A^i = 0,\qquad A^{ij} = k_1 g^{ij} + k_2 B^{ij}$$

$$A^{ijk} = k_1 E^{ijk} + k_2 B^i{}_\alpha E^{\alpha jk} + k_3 E^{ij\alpha} B^k{}_\alpha,\qquad A^{ijkl} = A^{ijkl}\ (m\cdot\infty : m)$$

Class $\infty : m$ $(g,\ B = e_3^2,\ \Omega = e_1 e_2 - e_2 e_1)$

$$A^i = 0,\qquad A^{ij} = k_1 g^{ij} + k_2 B^{ij} + k_3 \Omega^{ij},\qquad A^{ijk} = 0$$

$$A^{ijkl} = A^{ijkl}\ (m\cdot\infty : m) + k_{11} g^{ij} \Omega^{kl} + k_{12} g^{ik} \Omega^{jl} + k_{13} g^{il} \Omega^{jk} + k_{14} g^{kl} \Omega^{ij} + k_{15} g^{jl} \Omega^{ik} + k_{16} g^{jk} \Omega^{il} + k_{17} B^{ij} \Omega^{kl} + k_{18} B^{ik} \Omega^{jl} + k_{19} \Omega^{ij} B^{kl}$$

Class $\infty\cdot m$ $(g,\ b = e_3)$

$$A^i = k b^i,\quad A^{ij} = k_1 g^{ij} + k_2 b^i b^j,$$

$$A^{ijk} = k_1 g^{ij} b^k + k_2 g^{ik} b^j + k_3 g^{jk} b^i + k_4 b^i b^j b^k$$

$$A^{ijkl} = A^{ijkl}\ (\infty / \infty\cdot m) + k_4 g^{ij} b^k b^l + k_5 g^{ik} b^j b^l + k_6 g^{il} b^j b^k + k_7 g^{kl} b^i b^j + k_8 g^{jl} b^i b^k + k_9 g^{jk} b^i b^l + k_{10} b^i b^j b^k b^l$$

Class ∞ $(g,\ b = e_3,\ E)$

$$A^i = k b^i,\qquad A^{ij} = k_1 g^{ij} + k_2 b^i b^j + k_3 E^{ij\alpha} b_\alpha$$

$$A^{ijk} = k_1 g^{ij} b^k + k_2 g^{ik} b^j + k_3 g^{jk} b^i + k_4 b^i b^j b^k + k_5 \Omega^{ij} b^k + k_6 \Omega^{ik} b^j + k_7 \Omega^{jk} b^i$$

$$A^{ijkl} = A^{ijkl}\ (\infty\cdot m) + k_{11} g^{ij} \Omega^{kl} + k_{12} g^{ik} \Omega^{jl} + k_{13} g^{il} \Omega^{jk} + k_{14} g^{kl} \Omega^{ji} + k_{15} g^{jl} \Omega^{ik} + k_{16} g^{jk} \Omega^{il} + k_{17} b^i b^j \Omega^{kl} + k_{18} b^i b^k \Omega^{jl} + k_{19} \Omega^{ij} b^k b^l \qquad (\Omega^{ij} = E^{ij\alpha} b_\alpha)$$

The cubic system

Class $\bar{6} / 4$ (O_h)

$$A^i = 0,\quad A^{ij} = k g^{ij},\quad A^{ijk} = 0,\quad A^{ijkl} = A^{ijkl}\ (\infty / \infty\cdot m) + k_4 O_h{}^{ijkl}$$

Class $3 / 4$ $(O_h,\ E)$

$$A^i = 0,\qquad A^{ij} = k g^{ij},\qquad A^{ijk} = k E^{ijk},\qquad A^{ijkl} = A^{ijkl}\ (\bar{6} / 4)$$

Class $3 / \bar{4}$ $(g,\ T_d)$

$$A^i = 0,\qquad A^{ij} = k g^{ij},\qquad A^{ijk} = k T_d{}^{ijk},\qquad A^{ijkl} = A^{ijkl}\ (\bar{6} / 4)$$

Class $3 / 2$ $(g,\ E,\ T_d)$ or $(T_h,\ E)$

$$A^i = 0,\qquad A^{ij} = k g^{ij},\qquad A^{ijk} = k_1 E^{ijk} + k_2 T_d{}^{ijk}$$

$$A^{ijkl} = A^{ijkl}\ (\bar{6} / 4) + k_5 T_h{}^{ijkl} + k_6 T_h{}^{iljk} + k_7 T_h{}^{ikjl}$$

Class $\bar{6}/2$ (T_h)

$$A^i = 0, \quad A^{ij} = kg^{ij}, \quad A^{ijk} = 0 \quad A^{ijkl} = A^{ijkl}\,(3/2)$$

The tetragonal system

Class $m \cdot 4 : m$ $(O_h, B = e_3^2)$

$$A^i = 0, \qquad A^{ij} = A^{ij}\,(m \cdot \infty : m) = k_1 g^{ij} + k_2 B^{ij}$$
$$A^{ijk} = 0, \qquad A^{ijkl} = A^{ijkl}\,(m \cdot \infty : m) + k_{11} O_h^{\ ijkl}$$

Class $\bar{4} \cdot m$ $(g, T_d, B = e_3^2)$

$$A^i = 0, \qquad A^{ij} = k_1 g^{ij} + k_2 B^{ij}$$
$$A^{ijk} = k_1 T_d^{\ ijk} + k_2 T_d^{\ ij\alpha} B^{k\,\cdot}_{\ \cdot\,\alpha} + k_3 T_d^{\ ik\alpha} B^{j\,\cdot}_{\ \cdot\,\alpha}, \qquad A^{ijkl} = A^{ijkl}\,(m \cdot 4 : m)$$

Class $4 : 2$ $(O_h, B = e_3^2, E)$

$$A^i = 0, \quad A^{ij} = k_1 g^{ij} + k_2 B^{ij}, \quad A^{ijk} = A^{ijk}\,(\infty : 2), \quad A^{ijkl} = A^{ijkl}\,(m \cdot 4 : m)$$

Class $4 : m$ $(O_h, \Omega = e_1 e_2 - e_2 e_1, B = e_3^2)$

$$A^i = 0, \quad A^{ij} = A^{ij}\,(\infty : m), \quad A^{ijk} = 0$$
$$A^{ijkl} = A^{ijkl}\,(\infty : m) + k_{20} O_h^{\ ijkl} + k_{21} O_h^{\ jkl\alpha}\, \Omega^{i\,\cdot}_{\ \cdot\,\alpha}$$

Class $\bar{4}$ $(g, T_d, \Omega = e_1 e_2 - e_2 e_1, B = e_3^2)$

$$A^i = 0, \quad A^{ij} = A^{ij}\,(\infty : m), \quad A^{ijk} = A^{ijk}\,(\bar{4} \cdot m) + k_4 T_d^{\ ij\alpha} \Omega^{k\,\cdot}_{\ \cdot\,\alpha} +$$
$$+ k_5 \Omega^{i\,\cdot}_{\ \cdot\,\alpha} T_d^{\ \alpha jk} + k_6 \Omega^{i\,\cdot}_{\ \cdot\,\alpha} T_d^{\ \alpha j \beta} B^{k\,\cdot}_{\ \cdot\,\beta}, \qquad A^{ijkl} = A^{ijkl}\,(4 : m)$$

Class $4 \cdot m$ $(O_h, b = e_3)$

$$A^i = kb^i, \quad A^{ij} = k_1 g^{ij} + k_2 b^i b^j, \quad A^{ijk} = A^{ijk}\,(\infty \cdot m), \quad A^{ijkl} = A^{ijkl}\,(m \cdot 4 : m)$$

Class 4 $(O_h, b = e_3, E)$

$$A^i = kb^i, \qquad A^{ij} = k_1 g^{ij} + k_2 b^i b^j + k_3 \Omega^{ij} \qquad (\Omega^{ij} = E^{ij\alpha} b_\alpha)$$
$$A^{ijk} = A^{ijk}\,(\infty), \qquad A^{ijkl} = A^{ijkl}(\infty) + k_{20} O_h^{\ ijkl} + k_{21} O_h^{\ jkl\alpha} \Omega^{i\,\cdot}_{\ \cdot\,\alpha}$$

The hexagonal system

Class $m \cdot 6 : m$ $(D_{6h}, B = e_3^2)$

$$A^i = 0, \quad A^{ij} = k_1 g^{ij} + k_2 B^{ij}, \quad A^{ijk} = 0, \quad A^{ijkl} = A^{ijkl}\,(m \cdot \infty : m)$$

Class $m \cdot 3 : m$ $(D_{3h}, B = e_3^2)$

$$A^i = 0, \quad A^{ij} = k_1 g^{ij} + k_2 B^{ij}, \quad A^{ijk} = k D_{3h}^{\ ijk}, \quad A^{ijkl} = A^{ijkl}\,(m \cdot \infty : m)$$

Class $6 : 2$ $(D_{6h}, B = e_3^2, E)$

$$A^i = 0, \quad A^{ij} = k_1 g^{ij} + k_2 B^{ij}, \quad A^{ijk} = A^{ijk}\,(\infty : 2), \quad A^{ijkl} = A^{ijkl}\,(m \cdot \infty : m)$$

Class $6:m$ $(D_{6h},\ B = e_3^2,\ \Omega = e_1e_2 - e_2e_1)$

$$A^i = 0, \quad A^{ij} = A^{ij}(\infty : m), \quad A^{ijk} = 0, \quad A^{ijkl} = A^{ijkl}(\infty : m)$$

Class $3:m$ $(D_{3h},\ B = e_3^2,\ \Omega = e_1e_2 - e_2e_1)$

$$A^i = 0,\ A^{ij} = A^{ij}(\infty : m), \quad A^{ijk} = k_1 D_{3h}{}^{ijk} + k_2 D_{3h}{}^{ij\alpha}\Omega^{k\,\cdot}_{\,\cdot\,\alpha}, \quad A^{ijkl} = A^{ijkl}(\infty : m)$$

Class $6\cdot m$ $(D_{6h},\ b = e_3)$

$$A^i = kb^i, \quad A^{ij} = k_1 g^{ij} + k_2 b^i b^j, \quad A^{ijk} = A^{ijk}(\infty\cdot m), \quad A^{ijkl} = A^{ijkl}(\infty\cdot m)$$

Class 6 $(D_{6h},\ b = e_3,\ E)$

$$A^i = kb^i, \quad A^{ij} = A^{ij}(\infty), \quad A^{ijk} = A^{ijk}(\infty), \quad A^{ijkl} = A^{ijkl}(\infty)$$

The trigonal system

Class $\bar{6}\cdot m$ $(D_{3d},\ B = e_3^2)$

$$A^i = 0, \qquad A^{ij} = k_1 g^{ij} + k_2 B^{ij}, \qquad A^{ijk} = 0$$

$$A^{ijkl} = A^{ijkl}(m\cdot\infty : m) + k_{11} D_{3d}{}^{ijkl} + k_{12} D_{3d}{}^{jikl} + k_{13} D_{3d}{}^{kijl} + k_{14} D_{3d}{}^{lijk}$$

Class $3:2$ $(D_{3h},\ B = e_3^2,\ E)$

$$A^i = 0, \qquad A^{ij} = k_1 g^{ij} + k_2 B^{ij}, \qquad A^{ijk} = A^{ijk}(\infty : 2) + k_4 D_{3h}^{ijk}$$

$$A^{ijkl} = A^{ijkl}(m\cdot\infty : m) + k_{11} D_{3h}{}^{ij\alpha} E^{\,\cdot\,k\,l}_{\alpha\,\cdot\,\cdot} + k_{12} E^{\alpha ij} D_{3h}{}^{k\,l\,\cdot}_{\,\cdot\,\cdot\,\alpha} + k_{13} E^{\alpha ik} D_{3h}{}^{j\,l\,\cdot}_{\,\cdot\,\cdot\,\alpha} + k_{14} E^{kj} D_{3h}{}^{i\,l\,\cdot}_{\,\cdot\,\cdot\,\alpha}$$

Class $\bar{6}$ $(D_{3d},\ B = e_3^2,\ \Omega = e_1e_2 - e_2e_1)$

$$A^i = 0, \qquad A^{ij} = A^{ij}(\infty : m), \qquad A^{ijk} = 0$$

$$A^{ijkl} = A^{ijkl}(\infty : m) + k_{20} D_{3d}{}^{ijkl} + k_{21} D_{3d}{}^{jikl} + k_{22} D_{3d}{}^{kijl} + k_{23} D_{3d}{}^{lijk} + k_{24} D_{3d}{}^{ijk\alpha}\Omega^{l\,\cdot}_{\,\cdot\,\alpha} + k_{25} D_{3d}{}^{jik\alpha}\Omega^{l\,\cdot}_{\,\cdot\,\alpha} + k_{26} D_{3d}{}^{kij\alpha}\Omega^{l\,\cdot}_{\,\cdot\,\alpha} + k_{27} D_{3d}{}^{lij\alpha}\Omega^{k\,\cdot}_{\,\cdot\,\alpha}$$

Class $3\cdot m$ $(D_{3h},\ b = e_3)$

$$A^i = kb^i, \qquad A^{ij} = k_1 g^{ij} + k_2 b^i b^j, \qquad A^{ijk} = A^{ijk}(\infty\cdot m) + k_5 D_{3h}{}^{ijk}$$

$$A^{ijkl} = A^{ijkl}(\infty\cdot m) + k_{11} D_{3h}{}^{ijk} b^l + k_{12} D_{3h}{}^{ijl} b^k + k_{13} D_{3h}{}^{ikl} b^j + k_{14} D_{3h}{}^{klj} b^i$$

Class 3 $(D_{3h},\ b = e_3,\ E)$

$$A^i = kb^i, \qquad A^{ij} = A^{ij}(\infty)$$

$$A^{ijk} = A^{ijk}(\infty) + k_8 D_{3h}{}^{ijk} + k_9 D_{3h}{}^{ij\alpha}\Omega^{k\,\cdot}_{\,\cdot\,\alpha}, \qquad A^{ijkl} = A^{ijkl}(\bar{6})$$

The rhombic system

Class $m\cdot 2:m$ (D_{2h})

$$A^{i}=0,\quad A^{ij}=k_1 g^{ij}+k_2 D_{2h}{}^{ij}+k_3 D_{2h}{}^{i\alpha} D_{2h}{}_{\alpha\cdot}^{\cdot j}\qquad \text{(Cayley-Hamilton formula)}$$

$$A^{ijk}=0,\quad A^{ijkl}=k_1 g^{ij}g^{kl}+k_2 g^{ik}g^{jl}+k_3 g^{il}g^{jk}+k_4 g^{ij}D_{2h}{}^{kl}+k_5 g^{ik}D_{2h}{}^{jl}+$$
$$+k_6 g^{il}D_{2h}{}^{jk}+k_7 D_{2h}{}^{ij}g^{kl}+k_8 D_{2h}{}^{ik}g^{jl}+k_9 D_{2h}{}^{il}g^{jk}+k_{10}g^{ij}M^{kl}+$$
$$+k_{11}g^{ik}M^{jl}+k_{12}g^{il}M^{jl}+k_{13}M^{ij}g^{kl}+k_{14}M^{ik}g^{jl}+k_{15}M^{il}g^{jk}+k_{16}D_{2h}{}^{ij}D_{2h}{}^{kl}+$$
$$+k_{17}D_{2h}{}^{il}M^{jk}+k_{18}D_{2h}{}^{ij}M^{kl}+k_{19}D_{2h}{}^{ik}M^{jl}+k_{20}M^{ij}D_{2h}{}^{kl}+k_{21}M^{ij}M^{kl}$$
$$(M^{ij}=D_{2h}{}^{i\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot j})$$

Class $2:2$ $(D_{2h},\ E)$

$$A^{i}=0,\quad A^{ij}=A^{ij}\ (m\cdot 2:m)$$
$$A^{ijk}=k_1E^{ijk}+k_2E^{ij\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot k}+k_3E^{ik\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot j}+k_4E^{ij\alpha}M^{k\cdot}_{\cdot\alpha}+$$
$$+k_5E^{ik\alpha}M^{j\cdot}_{\cdot\alpha}+k_6D_{2h}{}^{i\alpha}E^{\cdot j\cdot}_{\alpha\cdot\beta}M^{\beta k},\quad A^{ijkl}=A^{ijkl}\,(m\cdot 2:m)$$

Class $2\cdot m$ $(D_{2h},\ \mathbf{b}=\mathbf{e}_3)$

$$A^{i}=kb^{i},\qquad A^{ij}=A^{ij}\,(m\cdot 2:m)=k_1g^{ij}+k_2b^ib^j+k_3D_{2h}{}^{ij}$$
$$A^{ijk}=k_1g^{ij}b^k+k_2g^{ik}b^j+k_3g^{kj}b^i+k_4b^ib^jb^k+k_5D_{2h}{}^{ij}b^k+k_6D_{2h}{}^{ik}b^j+k_7D_{2h}{}^{kj}b^i$$
$$A^{ijkl}=A^{ijkl}\,(m\cdot 2:m)$$

The monoclinic system

Class $2:m$ $(D_{2h},\ \Omega=\mathbf{e}_1\mathbf{e}_2-\mathbf{e}_2\mathbf{e}_1)$

$$A^{i}=0,\quad A^{ij}=A^{ij}\,(m\cdot 2:m)+k_4\Omega^{ij}+k_5\Omega^{i\alpha}D_{2h}{}^{j\cdot}_{\cdot\alpha},\quad A^{ijk}=0$$
$$A^{ijkl}=A^{ijkl}\,(m\cdot 2:m)+k_{22}g^{ij}\Omega^{kl}+k_{23}g^{ik}\Omega^{jl}+k_{24}g^{il}\Omega^{jk}+k_{25}g^{kl}\Omega^{ij}+k_{26}g^{jl}\Omega^{ik}+$$
$$+k_{27}g^{jk}\Omega^{il}+k_{28}g^{ij}\Omega^{k\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot l}+k_{29}g^{ik}\Omega^{j\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot l}+k_{30}g^{il}\Omega^{j\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot k}+k_{31}g^{kl}\Omega^{i\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot j}+$$
$$+k_{32}g^{jl}\Omega^{i\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot k}+k_{33}g^{jk}\Omega^{i\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot l}+k_{34}D_{2h}{}^{ij}\Omega^{kl}+k_{35}D_{2h}{}^{ik}\Omega^{jl}+k_{36}D_{2h}{}^{kl}\Omega^{ij}+$$
$$+k_{37}D_{2h}{}^{ij}\Omega^{k\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot l}+k_{38}D_{2h}{}^{il}\Omega^{j\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot k}+k_{39}D_{2h}{}^{kl}\Omega^{i\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot j}+$$
$$+k_{40}M^{kl}\Omega^{ij}+k_{41}M^{ij}\Omega^{k\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot l}$$

Class 2 $(D_{2h},\ E,\ \mathbf{b}=\mathbf{e}_3)$

$$A^{i}=kb^{i},\qquad A^{ij}=A^{ij}\ (2:m)$$
$$A^{ijk}=k_1g^{ij}b^k+k_2g^{ik}b^j+k_3g^{jk}b^i+k_4b^ib^jb^k+k_5D_{2h}{}^{ij}b^k+k_6D_{2h}{}^{ik}b^j+k_7D_{2h}{}^{kj}b^i+$$
$$+k_8\Omega^{ij}b^k+k_9\Omega^{ik}b^j+k_{10}\Omega^{kj}b^i+k_{11}\Omega^{i\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot j}b^k+k_{12}\Omega^{i\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot k}b^j+k_{13}\Omega^{k\alpha}D_{2h}{}_{\alpha\cdot}^{\cdot j}b^i$$
$$A^{ijkl}=A^{ijkl}\ (2:m)$$

Class m $(D_{2h},\ \mathbf{b} = \mathbf{e}_1,\ \mathbf{c} = \mathbf{e}_2)$

$$A^i = k_1 b^i + k_2 c^i, \qquad A^{ij} = k_1 g^{ij} + k_2 b^i b^j + k_3 c^i c^j + k_4 b^i c^j + k_5 c^i b^j$$

$$A^{ijk} = k_1 g^{ij} b^k + k_2 g^{ik} b^j + k_3 g^{jk} b^i + k_4 b^i b^j b^k + k_5 g^{ij} c^k + k_6 g^{ik} c^j + k_7 g^{jk} c^i + \\ + k_8 c^i b^j b^k + k_9 b^i c^j b^k + k_{10} b^i b^j c^k + k_{11} b^i c^j c^k + k_{12} c^i b^j c^k + k_{13} c^i c^j b^k + k_{14} c^i c^j c^k$$

$$A^{ijkl} = A^{ijkl}\,(2:m) = k_1 g^{ij} g^{kl} + k_2 g^{ik} g^{jl} + k_3 g^{il} g^{jk} + k_4 g^{ij} b^k b^l + k_5 g^{ik} b^j b^l + \\ + k_6 g^{il} b^j b^k + k_7 b^i b^j g^{kl} + k_8 b^i b^k g^{jl} + k_9 b^i b^l g^{jk} + k_{10} g^{ij} b^k c^l + k_{11} g^{ik} b^j c^l + \\ + k_{12} g^{il} b^j c^k + k_{13} g^{kl} b^i c^j + k_{14} g^{jl} b^i c^k + k_{15} g^{jk} b^i c^l + k_{16} g^{ij} c^k b^l + k_{17} g^{ik} c^j b^l + \\ + k_{18} g^{il} c^j b^k + k_{19} g^{kl} c^i b^j + k_{20} g^{jl} c^i b^k + k_{21} g^{jk} c^i b^l + k_{22} c^i b^j b^k b^l + k_{23} b^i c^j b^k b^l + \\ + k_{24} b^i b^j c^k b^l + k_{25} b^i b^j b^k c^l + k_{26} g^{ij} c^k c^l + k_{27} g^{ik} c^j c^l + k_{28} g^{il} c^j c^k + k_{29} c^k c^l c^i c^j + \\ + k_{30} b^i b^j b^k b^l + k_{31} g^{jk} c^i c^l + k_{32} b^i b^j c^k c^l + k_{33} b^i b^k c^j c^l + k_{34} b^i b^l c^j c^k + k_{35} c^i c^j b^k b^l + \\ + k_{36} c^i c^k b^j b^l + k_{37} c^i c^l b^j b^k + k_{38} b^i c^j c^k c^l + k_{39} c^i b^j c^k c^l + k_{40} c^j c^i b^k c^l + k_{41} c^i c^j c^k b^l$$

If $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3{}^2$ are taken as the defining tensors instead of D_{2h}, $\mathbf{e}_1$, $\mathbf{e}_2$, the last formula for fourth-order tensors may be replaced by the formula

$$A_4 = k^{ijkl} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l + k^{\alpha\beta 33} \mathbf{e}_\alpha \mathbf{e}_\beta \mathbf{e}_3 \mathbf{e}_3 + k^{\alpha 33\beta} \mathbf{e}_\alpha \mathbf{e}_3 \mathbf{e}_3 \mathbf{e}_\beta + k^{33\alpha\beta} \mathbf{e}_3 \mathbf{e}_3 \mathbf{e}_\alpha \mathbf{e}_\beta + \\ + k^{3\alpha 3\beta} \mathbf{e}_3 \mathbf{e}_\alpha \mathbf{e}_3 \mathbf{e}_\beta + k^{3\alpha\beta 3} \mathbf{e}_3 \mathbf{e}_\alpha \mathbf{e}_\beta \mathbf{e}_3 + k^{\alpha 3\beta 3} \mathbf{e}_\alpha \mathbf{e}_3 \mathbf{e}_\beta \mathbf{e}_3 + k^{3333} \mathbf{e}_3 \mathbf{e}_3 \mathbf{e}_3 \mathbf{e}_3 \qquad (*)$$

where the summation is carried out with respect to the indices i, j, k, l, α, β, which take on only the values 1 and 2. A simple calculation shows that there are 41 terms in this formula; their linear independence is immediately apparent.

It is not difficult to see that for tensors of even order, in particular, for fourth-order tensors referring to the classes $2:m$, 2 and m of the monoclinic system, the corresponding tensor parameters may be replaced by the same system of tensors, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3{}^2$. The same formulas may, therefore, be used. Thus, for all classes of the monoclinic system the formula (*) is applicable to fourth-order tensors.

It is also easy to see that the fourth-order tensors for the rhombic system with 21 linearly independent terms can be obtained from the formula (*) in which the terms with $i = j$, $k = l$; $i = k$, $j = l$; $i = l$, $j = k$ and $\alpha = \beta$ should be taken.

Thus, it is clear that in the construction of general formulas for tensor functions it is sometimes advantageous to change the original basis of arguments suitably for the particular cases at hand.

The triclinic system

Class $\bar{2}$ (C_i)

$$A^i = 0, \quad A^{ij} \text{ is the most general case with nine components}$$

$$A^{ijk} = 0, \quad A^{ijkl} \text{ is the most general case with 81 components}$$

Class 1 $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$

All tensors have the most general form if symmetries are absent

5. Tensor functions for oriented media and crystals with additional tensor arguments. We shall now assume that, besides the tensors which specify the geometric properties of oriented media or crystals, there are other tensors among the defining quantities or independent arguments. It is apparent that in this case the symmetry groups of the set of defining parametric tensors are suitable groups or subgroups of oriented media or crystals. Subgroups which differ from the crystallographic groups can arise only when considering oriented media. If other tensors are adjoined to those which determine a crystal symmetry, either some crystal symmetry group will be obtained again or the symmetry group will reduce to the identity transformation.

All subgroups of a given crystal symmetry group are contained among the 32 crystal groups. Therefore, upon addition of other tensors to those which specify the symmetry of the crystal, the symmetry group of the new set of arguments will also belong to one of the 32 crystal groups.

A decrease in the number of linearly independent components of the tensors defined in the general case can occur only in the presence of some corresponding symmetry. It is apparent that simplifications will take place in the case of crystals when the set of defining parameters admits a nontrivial symmetry group.

After the determination of the type of crystal symmetry group which is appropriate for a set of tensor arguments, one of the formulas of Section 4 can be used to determine the structure of the components of the tensor function which has been defined. Thus, it is possible to use the formulas of Section 4 to determine the structure of tensor functions for crystals in the general case. To ascertain the nature of the appropriate formulas it is first necessary to investigate the symmetry properties of the set of given arguments. For crystals this is equivalent to representing the defining tensors in terms of the set of tensors which characterize the crystal classes, as indicated in the table.

The argument given above permits us to analyze a large number of special cases easily, when the supplementary tensors are special or have a special form in the crystallographic axes. When additional tensors are present the scalars k_s are, in the general case, functions of the common invariants of the supplementary tensors and the tensors which specify the symmetry of the oriented media or crystals.

Supplementary tensors can give rise to variable simultaneous invariants. Generally the number of functionally independent invariants is equal to the number of functionally independent components of the variable tensors. In certain special cases the number of functionally independent components can be smaller.

It is possible to select the scalar invariants ω_i (in terms of which the k_s are defined) so that they retain their values for the different variable tensors which are equivalent from the point of view of symmetry of oriented media or crystals. These arguments, which are determined in a fixed coordinate system, may differ from the invariants Ω_i for arbitrary coordinate transformations but coincide with them ($\omega_i = \Omega_i$) in the given fixed coordinate system.

6. On the Riemannian curvature tensor and a generalization of Schur's theorem. The theory which has been developed above is directly related to all mathematical and physical laws which are formulated as vector or tensor equations and which, to some extent, are connected with geometric symmetry properties.

There are a great many important applications; we indicate as examples Hooke's law for oriented media and crystals, piezoelectric and optical effects, etc.

As one example we shall consider the Christoffel-Riemann curvature tensor R_{ijkl}. As is known [28] this tensor is antisymmetric with respect to interchange of the indices i and j or the indices k and l, and is symmetric with respect to interchange of the pairs of indices ij and kl. In the case of three-dimensional space there are only six independent components of R_{ijkl}, which may take on arbitrary independent values. These six components determine the six components of the symmetric second-order tensor K^{mn}, which may be introduced by the formula

$$K^{mn} = E^{ijm}E^{kln}R_{ijkl} \tag{6.1}$$

From this, we have

$$R_{ijkl} = \frac{1}{4}\ E_{ijm}E_{kln}K^{mn} \tag{6.2}$$

As is well known [31], the components of the curvature tensor satisfy the Bianchi identity

$$\nabla_r R_{ijmn} + \nabla_m R_{ijnr} + \nabla_n R_{ijrm} = 0$$

where the indices m, n, r are all different and ∇_K is the notation for covariant differentiation with respect to the coordinate x^K. It may easily be seen that Bianchi's identity is equivalent to the following identity in the components of the tensor K^{mn}:

$$\nabla_\alpha K^{m\alpha} = 0 \tag{6.3}$$

If the curvature tensor admits a symmetry of some type at points of the Riemannian space, then on the basis of the theory developed above it is easy to write out the general formulas which determine the components of R_{ijkl} and K^{mn} in terms of the tensors which specify the corresponding symmetry group.

For instance, for symmetries of the type of the oriented media the following formulas are valid:

for the symmetry $\infty/\infty \times m$ and ∞/∞

$$K^{mn} = kg^{mn} \tag{6.4}$$

for the symmetry $\infty \times m$, $m \times \infty : m$, $\infty : 2$, $\infty : m$, ∞

$$K^{mn} = kg^{mn} + k_1 b^m b^n \tag{6.5}$$

where b^m are the components of the unit vector directed along the axis of symmetry.

Analogous formulas can be written in any case when the components of the tensor K^{mn} admit any finite symmetry group. For instance, for symmetry corresponding to any one of the five classes of the cubic system we have:

$$K^{mn} = kg^{mn} \tag{6.6}$$

Therefore, in this case the tensor K^{mn} is spherical, just as in the case of complete isotropy. Corresponding formulas follow from (6.2) and (6.4) to (6.6) for the components of the tensor R_{ijkl}.

From (6.4) and Bianchi's identity (6.3), we have

$$g^{m\alpha} \nabla_\alpha k = 0 \tag{6.7}$$

The equation (6.7) expresses a well-known theorem of Schur. According to Schur's theorem, isotropy of the curvature tensor at each point implies the constancy of the curvature in the whole space. Indeed, we obtain

$$k = \text{const}$$

from (6.7).

A generalization of Schur's theorem is contained in the proof given above. This generalization consists of the fact that it is not necessary to require complete isotropy of the curvature at each point of the space for Schur's theorem to hold. It is sufficient that at each point the symmetry conditions of the group 3/2 be satisfied, i.e. that the components of the tensors K^{mn} and R_{ijkl} be invariant under the 12 transformations of the symmetry group 3/2.

If the curvature is determined at each point by constant, collinear vectors b^i, then Bianchi's identity gives:

$$\nabla^{\lambda} k + b^{\lambda} b^{\mu} \nabla_{\mu} k_1 = 0 \tag{6.8}$$

Equations (6.8) are a system of equations imposed on the curvature for the corresponding Riemannian spaces.

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